AVERAGE OF THE INTERSECTION NUMBERS OF PAIRS OF CLOSED GEODESICS IN A SURFACE

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Abstract. For a compact surface with negative curvature X we show that the normalized average of the intersection numbers of pairs of closed geodesics of X is asymptotically equal to $1/(2\pi^2(g-1))$, where g is the genus of X.

1. Introduction

Let X be a compact hyperbolic surface, $\mathcal{G}$ be the set of closed geodesics of X and $\mathcal{G}_K$ be the set of closed geodesics of X of length at most K. We denote the number of elements of $\mathcal{G}_K$ by $N(K)$. It is a classical result of Margulis (see [Ma]) that the number $N(K)$ satisfies the asymptotic formula $N(K) \sim e^K/2K$ (read "$N(K)$ is asymptotically equal to $e^K/2K$"), as $K \to \infty$, i.e., the ratio of the two sides converges to one.

For $\alpha, \beta \in \mathcal{G}$ we denote by $i(\alpha,\beta)$ the (*geometric*) *intersection number of* α *and* β, that is the number of points of intersections of α and β. In particular, $i(\gamma,\gamma)$ is the number of self-intersections of a closed geodesic γ. Lalley showed in [La] that for $K \gg 0$, the number of self-intersections of most of the closed geodesics of length K is about $K^2/(2\pi^2(g-1))$, where g is the genus of X. In this paper we show that the normalized average of pairs of closed geodesics of lengths at most J and K is asymptotically equal to $1/(2\pi^2(g-1))$, as these lengths become arbitrarily large.

Theorem 1 (Herrera).
$$\frac{1}{N(J)N(K)} \sum_{(\alpha,\beta)\in\mathcal{G}_J\times\mathcal{G}_K} \frac{i(\alpha,\beta)}{l(\alpha)l(\beta)} \sim \frac{1}{2\pi^2(g-1)},$$
as $J, K \to \infty$.

As an immediate consequence, we get that the normalized average of the numbers of self-intersections of the closed geodesics of the surface of length at most K is also asymptotically equal to $1/(2\pi^2(g-1))$, as K grows to infinity.

Theorem 2 (Herrera).
$$\frac{1}{N(K)^2} \sum_{\alpha\in\mathcal{G}_K} \frac{i(\alpha,\alpha)}{l(\alpha)^2} \sim \frac{1}{2\pi^2(g-1)},$$
as $K \to \infty$.

In order to prove Theorem 1 we use the extension of the intersection number function to the intersection form function on the set $\mathcal{C}$ of current measures of the surface X introduced by Bonahon in [Bo1]. For a more detailed explanation see Section 2.2. Let $T^1(X)$ be the unit tangent bundle of X, Φ be the geodesic flow on $T^1(X)$ and E be the Whitney sum of $T^1(X) \oplus T^1(X)$ minus the diagonal $\triangle$. For

each closed geodesic γ there exists a unique measure $\widetilde{\mu}_\gamma$ which is transverse to the Φ-leaves of $T^1(X)$. For this measure $\widetilde{\mu}_\gamma$ Bonahon defined $\widehat{\mu}_{\gamma,t} = p_t^{-1}(\widetilde{\mu}_\gamma)$, where $p_t : E \to T^1(X)$ are given by $p_t((x, w_1, w_2)) = (x, w_t)$, for $t = 1, 2$; and showed that $(\widehat{\mu}_{\alpha,1} \times \widehat{\mu}_{\beta,2})(E) = i(\alpha, \beta)$, for $\alpha, \beta \in \mathcal{G}$.

One of the key points for the proof of Theorem 1 is to show that the size of the set of pairs of geodesics with intersection number greater than $1/(2\pi^2(g-1))$ decreases exponentially fast as the lengths of the geodesics grow. For this end, we consider the set

$$W(J, K, \epsilon) = \left\{(\alpha, \beta) \in \mathcal{G}_J \times \mathcal{G}_K : \left| \frac{(\widehat{\mu}_{\alpha,1} \times \widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} - \frac{1}{2\pi^2(g-1)} \right| < \epsilon \right\},$$

for positive real numbers J, K, ϵ.
The following theorem states that the size of the complement of the set $W(J, K, \epsilon)$ decreases exponentially fast as J and K become very large.

Theorem 3 (Herrera)**.** *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\frac{\#[(\mathcal{G}_J \times \mathcal{G}_K) \setminus W(J, K, \epsilon)]}{N(J)N(K)} = O(e^{-\delta K}),$$

for every $J, K \to \infty$.

The other key point for the proof of Theorem 1 is to find a bound for the intersection numbers of the pairs of closed geodesics of X. This was achieved by Basmajian in [Ba]. Here we give a bound different from the one given by Basmajian as well as a different proof.

Proposition 1 (Basmajian)**.** *Let $\alpha, \beta \in \mathcal{G}$ and $\varepsilon_0 = \operatorname{inj} X$, the injectivity radius of X. Then $i(\alpha, \beta) \leq \dfrac{4l(\alpha)l(\beta)}{\varepsilon_0^2}$.*

Proof. Let $\alpha : [0, l(\alpha)] \to X$ and $\beta : [0, l(\beta)] \to X$ be closed geodesics of X. Let $\bar{\alpha}$ be a subarc of α of length less than $\varepsilon_0/2$ for which $i(\bar{\alpha}, \beta)$ is the largest. Hence, $i(\alpha, \beta) \leq [l(\alpha)/(\varepsilon_0/2)]i(\bar{\alpha}, \beta)$. Let $\{x_1, x_2, \ldots, x_n\}$ be the ordered set of points of intersection of $\bar{\alpha}$ and β, where $\beta^{-1}(x_i) \leq \beta^{-1}(x_{i+1})$, for $1 \leq i \leq n-1$, and $n = i(\bar{\alpha}, \beta)$. Let β_t be the subarc of β from x_t to x_{t+1}, for $1 \leq t \leq n-1$ and β_n the subarc of β joining x_1 and x_n which does not intersect $\bar{\alpha}$. Similarly, let $\bar{\alpha}_t$ be the subarc of α from x_t to x_{t+1}, for $1 \leq t \leq n-1$, and $\bar{\alpha}_n$ be the subarc joining x_1 and x_n not intersecting $\bar{\alpha}$. Consider γ_t the concatenation of $\bar{\alpha}_t$ and β_t, for $1 \leq t \leq n$. Thus, γ_t is an essential loop of X, for $1 \leq t \leq n$. Hence, $\varepsilon_0 < l(\gamma_t) = l(\bar{\alpha}_t) + l(\beta_t) \leq \varepsilon_0/2 + l(\beta_t)$, which implies $\varepsilon_0/2 \leq l(\beta_t)$, for $1 \leq t \leq n$. Consequently, $n < l(\beta)/(\varepsilon_0/2)$. Otherwise, $l(\beta) > \sum_{t=1}^n l(\beta_t) \leq (n)l(\beta)/(\varepsilon_0/2) = l(\beta)$, which is a contradiction. Therefore,

$$i(\alpha, \beta) \leq \frac{l(\alpha)}{(\varepsilon_0/2)} i(\alpha', \beta) \leq \frac{l(\alpha)}{(\varepsilon_0/2)} \frac{l(\beta)}{(\varepsilon_0/2)} = \frac{4l(\alpha)l(\beta)}{\varepsilon_0^2}.$$

$\square$

Remark 1. *Proposition 1 implies $i(\alpha, \beta) \leq \dfrac{4JK}{\varepsilon_0^2}$, whenever $(\alpha, \beta) \in \mathcal{G}_J \times \mathcal{G}_K$.*

The outline of this paper is the following. Section 2 is the collection of definitions and results needed in the demonstrations of Theorems 8 and 3. Section 3 contains

the proofs of the Theorems 8, 3, 1 and 2. For detailed explanation of all the concepts used in this article, see [HaKa] and [Po].

2. Preliminars

2.1. **Tangent Bundles.** Let $T^1(X) = \{(x,v) \mid x \in X, v \in T_xX, \|v\| = 1\}$, the *unit tangent bundle of* X, $\Phi = \{\phi_t\}$ be the *geodesic flow of* $T^1(X)$ and $\mathcal{F}$ be the *foliation of* $T^1(X)$ *by* Φ*-orbits.*

Consider E the *Whitney sum of* $T^1(X) \oplus T^1(X)$ minus the following set $\triangle = \{(x,v,v) \mid x \in X, v \in T_xX\}$, that is, $E = \{(x,v,w) | (x,v),(x,w) \in T^1(X), v \neq w\}$. Let $\pi : T^1(X) \to X$ be defined by $\pi((x,v)) = x; p : E \to X$ be defined by $p((x,v,w)) = x$; and $p_1, p_2 : E \to T^1(X)$ be defined by $p_1((x,v,w)) = (x,v)$ and $p_2((x,v,w)) = (x,w)$, respectively.

Denote by $\mathfrak{M}$ the set of the Φ-invariant measures in $T^1(X)$ equipped with the weak*-topology and let $h(\mu)$ denote the *measure theoretic entropy of* Φ with respect to $\mu \in \mathfrak{M}$, and $h = \max_{\mu\in\mathfrak{M}} h(\mu)$. In [KaHa], Katok and Hasselblatt showed that for a compact hyperbolic surface there is a unique Φ-invariant measure in $T^1(X)$ with maximum entropy h.

Theorem 4. *For a compact hyperbolic surface X, the measure of maximal entropy in $\mathfrak{M}$ is the Liouville measure $\mathcal{L}$ of the unit tangent bundle $T^1(X)$.*

2.2. **Current Measures and Intersection Form.** For each $\mu \in \mathfrak{M}$, which is finite, there exists an associated transverse measure to $\mathcal{F}$, which we denote by $\widetilde{\mu}$. The set of all of these transverse measures equipped with the weak*-topology is known as the *space of current measures of* X and we denoted by $\mathcal{C}$. Each $\widetilde{\mu} \in \mathcal{C}$ is normalized by the requirement that (locally) $\mu = \widetilde{\mu} \times \mathrm{d}t$, where $\mathrm{d}t$ is the one-dimensional Lebesgue measure along orbits in $\mathcal{F}$. In addition, for $\widetilde{\mu} \in \mathcal{C}$ define the *length of* $\widetilde{\mu}$ by $L(\widetilde{\mu}) = \mu(T^1(X))$.

Consider the following foliations $\mathcal{F}_1 = p_1^{-1}(\mathcal{F})$ and $\mathcal{F}_2 = p_2^{-1}(\mathcal{F})$ of $T^1(X)$, for p_1 and p_2 as defined in Section 2.1. Furthermore, for $\widetilde{\mu}, \widetilde{\nu} \in \mathcal{C}$, define $\widehat{\mu}_1 = p_1^{-1}(\widetilde{\mu})$ and $\widehat{\nu}_2 = p_2^{-1}(\widetilde{\nu})$. Note that the new measures $\widehat{\mu}_1$ and $\widehat{\nu}_2$ are transverse to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The *intersection form of* $\widetilde{\mu}$ *and* $\widetilde{\nu}$, denoted by $I(\widetilde{\mu}, \widetilde{\nu})$, is defined as the total mass of E with respect to the product measure $\widehat{\mu}_1 \times \widehat{\nu}_2$, that is, $I(\widetilde{\mu}, \widetilde{\nu}) = \int_E \mathrm{d}(\widehat{\mu}_1 \times \widehat{\nu}_2) = (\widehat{\mu}_1 \times \widehat{\nu}_2)(E)$.

Bonahon showed in [Bo2] that the value of the intersection form at $(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}})$, where $\widetilde{\mathcal{L}}$ is the current measure corresponding to the Liouville measure $\mathcal{L}$ of the unit tangent bundle of the surface X depends only on its genus g.

Theorem 5 (Bonahon)**.** $I(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}) = \pi^2|\chi(X)| = 2\pi^2(g-1)$.

Now, for every $\gamma \in \mathcal{G}$ there exists a unique invariant measure μ_γ of total mass $l(\gamma)$, where $l(\gamma)$ is the length of the geodesic γ with respect to the hyperbolic metric of X. This measure μ_γ is supported on the orbit of γ. Let $\widetilde{\mu}_\gamma$ denote the corresponding transverse measure to the orbit foliation $\mathcal{F}$. These current measures are normalized to be finite sums of Dirac measures on the transverse sections.

By identifying the current measure $\widetilde{\mu}_\gamma$ with the corresponding geodesic γ, Bonahon showed in [Bo1] that the intersection form function I is a continuous extension of the intersection number function i. In order to simplify our notation, we write $\widehat{\mu}_{\gamma,n}$ instead of $\widehat{\mu_{\gamma}}_t$, for $t = 1, 2$.

Theorem 6 (Bonahon)**.** *The intersection form* $I : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ *is a continuous extension of the intersection number function* $i : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$. *In particular,* $I(\widetilde{\mu}_\alpha, \widetilde{\mu}_\beta) = \int_E \mathrm{d}(\widehat{\mu}_{\alpha,1} \times \widehat{\mu}_{\beta,2}) = i(\alpha, \beta)$.

3. Proofs of the main results

In the proof of the Theorem 3 we use the following result obtained by Kifer in [Ki], which states that the portion of "irregular" geodesics vanishes exponentially fast.

Theorem 7 (Kifer)**.** *Let* $\mathcal{U}$ *be an open neighbourhood of* $\mathcal{L}$ *in* $\mathfrak{M}$. *Then, there exists* $\delta > 0$ *such that*

$$\frac{1}{N(K)} \#\{\gamma \in \mathcal{G}_K : \mu_\gamma / l(\gamma) \notin \mathcal{U}\} = O(e^{-\delta K}),$$

as $K \to \infty$. *Moreover,* $\delta = \inf_{\nu \in \mathcal{U}^c} \{h - h(\nu)\}$.

Theorems 6 and 7 imply our first result, Theorem 3.

Proof of Theorem 3. Let $J, K, \epsilon > 0$ with $K \leq J$. Define $\mathfrak{f} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ by

$$\mathfrak{f}(\widetilde{\mu}, \widetilde{\nu}) = \frac{I(\widetilde{\mu}, \widetilde{\nu})}{L(\widetilde{\mu}) L(\widetilde{\nu})}.$$

The function $\mathfrak{f}$ is a continuous function because it is the quotient of two continuous functions, the intersection form function, which is continuous by Theorem 6, and the function that assigns the product of the lengths to a pair of current measures. As a result, the set $\mathcal{Z} = \mathfrak{f}^{-1}\left(\frac{1}{2\pi^2(g-1)} - \epsilon, \frac{1}{2\pi^2(g-1)} + \epsilon \right)$, the preimage of the ball of radius ϵ centered at $\mathfrak{f}(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}}) = \frac{I(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}})}{L(\widetilde{\mathcal{L}}) L(\widetilde{\mathcal{L}})} = \frac{2\pi^2(g-1)}{[2\pi^2(g-1)]^2} = \frac{1}{2\pi^2(g-1)}$, is an open subset of $\mathcal{C} \times \mathcal{C}$.
Note that $W(J, K, \epsilon) = \{(\alpha, \beta) \in \mathcal{G}_J \times \mathcal{G}_K : (\widetilde{\mu}_\alpha / l(\alpha), \widetilde{\mu}_\beta / l(\beta)) \in \mathcal{Z}\}$.
Since $\mathcal{Z}$ is an open set of the product topology of $\mathcal{C} \times \mathcal{C}$, there exist $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ open neighborhoods of $\widetilde{\mathcal{L}}$ such that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{Z}$.
Consider $U_J := \{\gamma \in \mathcal{G}_J : \widetilde{\mu}_\gamma / l(\gamma) \in \mathcal{U}\}$ and $V_K := \{\gamma \in \mathcal{G}_K : \widetilde{\mu}_\gamma / l(\gamma) \in \mathcal{V}\}$. Then, $U_J \times V_K \subseteq W(J, K, \epsilon)$, and by Theorem 7, there exist $\delta_1, \delta_2, R_1, R_2, C_1, C_2 > 0$ such that for $J \geq R_1$ and $K \geq R_2$,

$$\frac{1}{N(J)} \#\{\mathcal{G}_J \setminus U_J\} \leq \frac{C_1}{e^{\delta J}} \quad \text{and} \quad \frac{1}{N(K)} \#\{\mathcal{G}_K \setminus V_K\} \leq \frac{C_2}{e^{\delta K}}.$$

Thus, taking $R = \max\{R_1, R_2\}, C = C_1 + C_2 + C_1 C_2$ and $\delta = \min\{\delta_1, \delta_2\}$, we get

$$\begin{aligned}
\frac{\#[(\mathcal{G}_J \times \mathcal{G}_K) \setminus W(J, K, \epsilon)]}{N(J)N(K)} &\leq \frac{\#[(\mathcal{G}_J \times \mathcal{G}_K) \setminus (U_J \times V_K)]}{N(J)N(K)} \\
&= \frac{\#[\mathcal{G}_J \setminus U_J] \#[\mathcal{G}_K \setminus V_K]}{N(J)N(K)} + \frac{\#[\mathcal{G}_J \setminus U_J] \# V_K}{N(J)N(K)} + \frac{\#[\mathcal{G}_K \setminus V_K] \# U_J}{N(J)N(K)} \\
&\leq \frac{C_1 C_2}{e^{\delta_1 J} e^{\delta_2 K}} + \frac{C_1}{e^{\delta_1 J}} + \frac{C_2}{e^{\delta_2 K}} \leq \frac{C}{e^{\delta K}},
\end{aligned}$$

whenever $R \leq K \leq J$. Hence, we conclude the result of the theorem. $\square$

Theorems 1 and 2 are straightforward results from the following theorem.

Theorem 8 (Herrera)**.**

$$\frac{1}{N(J)N(K)} \sum_{(\alpha,\beta)\in\mathcal{G}_J\times\mathcal{G}_K} \frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} \sim \frac{1}{2\pi^2(g-1)},$$

as $J, K \to \infty$. Or equivalently,

$$\lim_{J,K\to\infty} \frac{2\pi^2(g-1)}{N(J)N(K)} \sum_{(\alpha,\beta)\in\mathcal{G}_J\times\mathcal{G}_K} \frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} = 1.$$

Proof. Let $J, K, \epsilon > 0$ with $K \leq J$. Define $W(J,K,\epsilon)$ as in Theorem 3. Then, there exist $C, R, \delta > 0$ such that

$$\frac{\#[(\mathcal{G}_J\times\mathcal{G}_K)\setminus W(J,K,\epsilon)]}{N(J)N(K)} \leq \frac{C}{e^{\delta K}},$$

whenever $R \leq K \leq J$.

We can take $R > 0$, so that $\frac{4JKC}{\varepsilon_0^2 e^{\delta K}} < \epsilon$ for all $R \leq K \leq J$, since $\frac{JK}{e^{\delta K}} \to 0$ as $J, K \to \infty$. Therefore,

$$\begin{aligned}
&\left|\frac{2\pi^2(g-1)}{N(J)N(K)} \sum_{(\alpha,\beta)\in\mathcal{G}_J\times\mathcal{G}_K} \frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} - 1\right| \\
&\quad= \frac{2\pi^2(g-1)}{N(J)N(K)} \sum_{(\alpha,\beta)\in W(J,K,\epsilon)} \left|\frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} - \frac{1}{2\pi^2(g-1)}\right| \\
&\quad\leq \frac{2\pi^2(g-1)}{N(J)N(K)} \left[\sum_{(\alpha,\beta)\in W(J,K,\epsilon)} \left|\frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} - \frac{1}{2\pi^2(g-1)}\right|\right. \\
&\qquad\qquad \left. + \sum_{(\alpha,\beta)\in(\mathcal{G}_J\times\mathcal{G}_K)\setminus W(J,K,\epsilon)} \left|\frac{(\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)}{l(\alpha)l(\beta)} - \frac{1}{2\pi^2(g-1)}\right|\right] \\
&\quad\leq \frac{2\pi^2(g-1)}{N(J)N(K)} \left[(\#[W(J,K,\epsilon)])\epsilon\right. \\
&\qquad\qquad \left. + \#[(\mathcal{G}_J\times\mathcal{G}_K)\setminus W(J,K,\epsilon)] \cdot \sup_{(\alpha,\beta)\in\mathcal{G}_J\times\mathcal{G}_K} |i(\alpha,\beta)-\epsilon|\right] \\
&\quad< \frac{2\pi^2(g-1)}{N(J)N(K)} \left[\epsilon + \#[(\mathcal{G}_J\times\mathcal{G}_K)\setminus W(J,K,\epsilon)] \frac{|JK-\epsilon|}{\varepsilon_0^2}\right] \\
&\quad< 2\pi^2(g-1)\left[\epsilon + \frac{C}{e^{\delta K}}\frac{4JK}{\varepsilon_0^2}\right] < 4\pi^2(g-1)\epsilon.
\end{aligned}$$

Given that ϵ was chosen arbitrarily, we conclude the result of the theorem. □

Proof of Theorems 1 and 2. By Theorem 6, $i(\alpha,\beta) = (\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\beta,2})(E)$ and $i(\alpha,\alpha) = (\widehat{\mu}_{\alpha,1}\times\widehat{\mu}_{\alpha,2})(E)$. Hence, we conclude the results by Theorem 8. □

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